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Hilbert's method for numerical solution of flow from a uniform channel over a shelf

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SUMMARY

A nonlinear problem for the flow from a uniform channel over a shelf has been solved, using conformal mapping and the Hilbert solution of a mixed-boundary-value problem in the upper half-plane. The solution in the gravity-affected case was found by numerical iteration; the nongravity solution was used as an initial approximation. The numerical solutions obtained have been compared with those of other authors. Favourable agreement with the results of experiments suggest that this method is effective in dealing with flow problems strongly influenced by gravity. Some difficulties of the computing, and some checking of the solution, are discussed.

1. Introduction

Much has been written about the problem of free streamlines. One problem of particular interest is the flow from an open uniform (rectangular) channel over a shelf under the influence of gravity. The configuration is simple, yet the effect of gravity precludes solutions in closed form.

The problem dates back to Leonardo da Vinci (see Rouse and Ince [6]). Experimental work was carried out in 1936 by Rouse, who found that the depth at the brink y_b, divided by the critical depth y_{cr}, was equal to 0.715. Southwell and Vaisey [9] used a relaxation method, obtaining the value 0.705. Hay and Markland [4] used an electrolytic plotting tank, obtaining the value 0.676.

Analytic work, using matched inner and outer asumptotic expansions, was carried out by Clarke [2] in 1965, who used the inverse of the Froude number as a small parameter. In 1979, Keller and Geer [3, 5], using a similar approach, considered more general flows, finding solutions valid for large Froude numbers, but disagreeing with observed values far downstream.

In 1979, Chow and Han [1] used a hodograph method which agreed with Rouse's experimental data, especially when the grid size was refined in the numerical treatment.

In this paper, the Hilbert method is used as a basis for the numerical solution of a two-dimensional problem. First, as customary, the physical and complex potential planes are mapped onto an auxiliary upper half-plane. Then nonlinear integral equations are found giving

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the solution of a mixed boundary value problem. Lastly, numerical solutions of those equations are obtained by iteration, using the flow in the absence of gravity as a starting solution.

The problem is formulated in Section 2, in which a set of integral equations is obtained, relating only the *boundary* values. In Section 3, infinite integrals are reduced to finite integrals by a suitable transformation. Sections 4 treats of the numerical scheme. In Section 5, the results are discussed and compared with those of Chow and Han [1], Clarke [2], Southwell and Vaisey [9] and the experimental work of Rouse [6].

2. Formulation of the problem

An inviscid, incompressible fluid flows over a horizontal surface until it falls over an edge under the influence of gravity. The flow is considered to be two-dimensional, steady and irrotational. Far upstream the fluid is of depth h and has a uniform horizontal velocity U, and gravity acts vertically downwards, see Fig. 1.

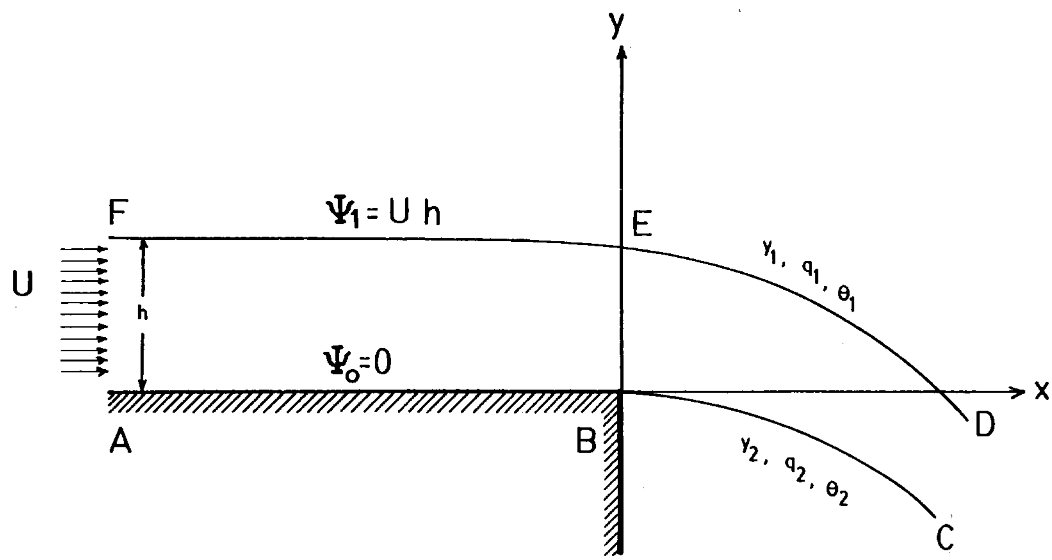

Figure 1. The physical plane, Z-plane, for a flow from uniform channel over shelf.

For convenience, we choose point B to be the origin in the z-plane, the x-axis from left to right and the y-axis upwards. The complex potential $w(z) = \phi(x,y) + i\psi(x,y)$ is an analytic function of z within the region of flow, with complex conjugate velocity

$$\frac{dw(z)}{dz} = u(x,y) - iv(x,y) = qe^{-i\theta}. \tag{2.1}$$

Along the upper free surface let q_1, y_1, θ_1 be the speed of the fluid, the vertical distance between a point on the free surface and some reference elevation, and the angle of inclination

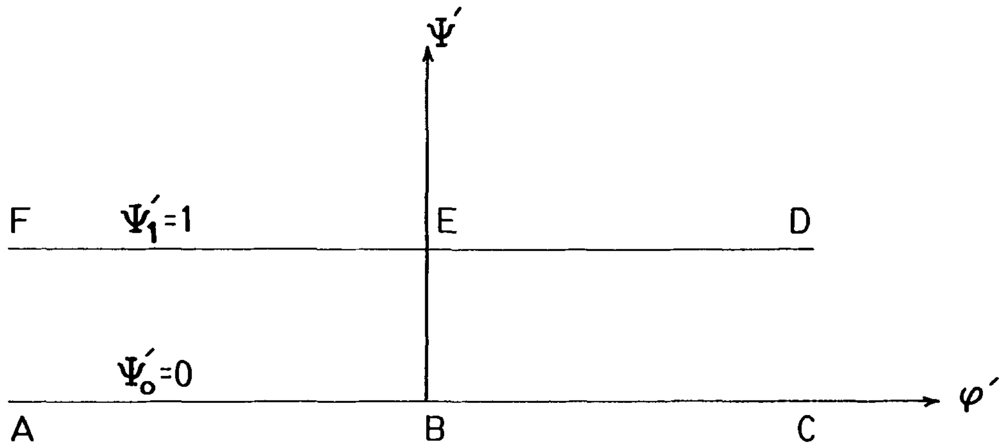

Figure 2. The normalized complex potential-plane, W′-plane, for a flow from uniform channel over shelf.

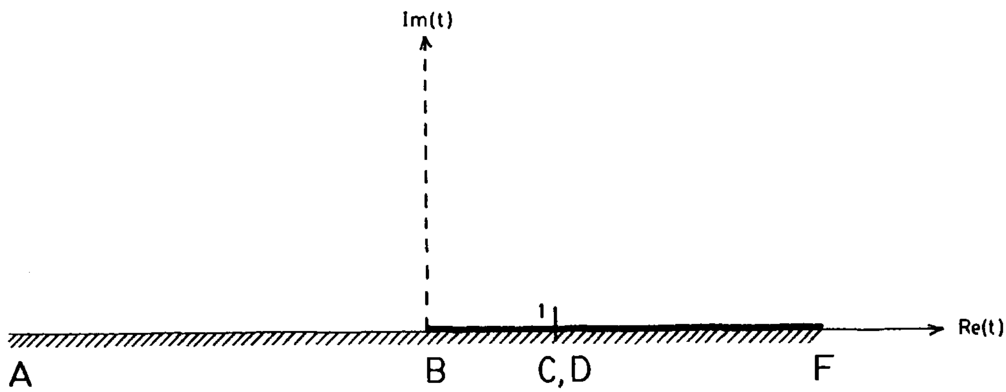

Figure 3. The upper half-plane, t-plane for a flow from uniform channel over shelf.

of the velocity with the horizontal, respectively; similarly for q_2, y_2 and θ_2 along the *lower* free surface.

Let the dimensionless variables z', q', and w' be:

$$z_i' = \frac{z_i}{h}, \qquad q_i' = \frac{q_i}{h}, \qquad w' = \frac{w}{\psi_1} \tag{2.2}$$

where $i = 1, 2$ and $\psi_1 = hU$.

In dimensionless form, the free-surface conditions along the upper and lower free surfaces, respectively, are

$$q_1'^2 + \frac{2}{F^2}(y_1' - 1) = 1,$$
$$q_2'^2 + \frac{2}{F^2}(y_2' - 1) = 1, \tag{2.3}$$

where F is the Froude number defined by

$$F = \frac{U}{\sqrt{gh}}. \tag{2.4}$$

Then (2.1) takes the form

$$\zeta = \frac{\mathrm{d}w'}{\mathrm{d}z'} = q' e^{-i\theta} \tag{2.5}$$

Let

$$\omega = \log \zeta = \log q' + i(-\theta) \tag{2.6}$$

Then

$$z' = \int e^{-\omega} \,\mathrm{d}w'. \tag{2.7}$$

Using the Schwartz-Christoffel transformation we map the region of flow in the w'-plane onto the upper half of an auxiliary t-plane, so that the following points correspond (see Figures 2 and 3)

$$\begin{aligned} &B\colon w' = 0, && t = 0;\\ &C,D\colon w' \to +\infty, && t = 1;\\ &A,F\colon w' \to -\infty, && t \to \infty. \end{aligned} \tag{2.9}$$

The mapping is

$$w'(t) = -\frac{1}{\pi}\log(1-t) \tag{2.10}$$

where

$$0 \leqslant \arg(1-t) \leqslant \pi. \tag{2.11}$$

For the second half of the problem, to express ω as a function of the single variable t, we introduce the Hilbert method for a mixed boundary-value problem in the upper half-plane, the general solution of which for an analytic function $Q(t)$ in the upper half-plane, is given by

$$Q(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{Im}[Q(u)]}{u-t} \mathrm{d}u + \sum_{j=0}^{n} A_j t^j, \tag{2.12}$$

where A_j are real constants.
Now

$$\begin{aligned} \mathrm{Im}(\omega(t)] &= 0, \qquad t < 0; \\ \mathrm{Re}[\omega(t)] &= \tfrac{1}{2} \log \left[1 - \frac{2}{F^2}(y_2' - 1 \right], \qquad 0 > t < 1; \\ \mathrm{Re}[\omega(t)] &= \tfrac{1}{2} \log \left[1 - \frac{2}{F^2}(y_1' - 1 \right], \qquad t > 1. \end{aligned} \tag{2.13}$$

Thus we know either the imaginary or real part of $\omega(t)$ along the real axis of the t-plane.

We then introduce an auxiliary function $H(t)$ such that $\mathrm{Im}[Q(t)]$, where $Q(t) = \omega(t)/H(t)$, is known at all points of the real axis. The general solution for $H(t)$ is

$$H(t) = \prod_j (t - b_j)^{\pm 1/2}, \tag{2.14}$$

where the b_j are real.

Song [8] has shown that the final solution is independent of the particular choice of $H(t)$. We choose

$$H(t) = -i\sqrt{t}, \qquad 0 \leqslant \arg t \leqslant \pi. \tag{2.15}$$

Using (2.13) and (2.15), we obtain

$$\mathrm{Im}[Q(t)] = \begin{cases} 0 & , \qquad t > 0 \\ \tfrac{1}{2} \left\{ \log \left[1 - \frac{2}{F^2}(y_2'(t) - 1) \right] \right\} t^{-1/2}, & \qquad 0 < t > 1 \\ \tfrac{1}{2} \log \left[1 - \frac{2}{F^2}(y_1'(t) - 1) \right] t^{-1/2}, & \qquad t > 1. \end{cases} \tag{2.16}$$

Applying the upstream condition, $A_j = 0$; $j = 0, 1, 2, \ldots, n$ in (2.12). Thus (2.12) takes the form

$$Q(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Im}[Q(u)]}{u-t} du. \tag{2.17}$$

Using (2.6), we obtain

$$\begin{aligned} Q(t) &= \frac{\omega(t)}{H(t)} = \frac{\log q'(t) + i(-\theta)}{H(t)} \\ &= U(t) + iV(t). \end{aligned} \tag{2.18}$$

An equivalent form of (2.17) is

$$U(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{V(u)}{u-t} du, \tag{2.19}$$

$$V(t) = \frac{-1}{\pi} \int_{-\infty}^{\infty} \frac{U(u)}{u-t} du. \tag{2.20}$$

Using (2.19) and (2.20), we obtain the following equations

$$\log q'(t) = \frac{-\sqrt{-t}}{\pi} \left\{ \int_0^1 \frac{\log q_2'(u)}{(u-t)\sqrt{u}} du + \int_1^{\infty} \frac{\log q_1'(u)}{(u-t)\sqrt{u}} du \right\}, \qquad t < 0; \tag{2.21}$$

$$\theta_2(t) = \frac{\sqrt{t}}{\pi} \left\{ ⨍_0^1 \frac{\log q_2'(u)}{(u-t)\sqrt{u}} du + \int_1^{\infty} \frac{\log q_1'(u)}{(u-t)\sqrt{u}} du \right\}, \qquad 0 < t < 1 \tag{2.22}$$

$$\theta_1(t) = \frac{\sqrt{t}}{\pi} \left\{ \int_0^1 \frac{\log q_2'(u)}{(u-t)\sqrt{u}} du + ⨍_1^{\infty} \frac{\log q_1'(u)}{(u-t)\sqrt{u}} du \right\}, \qquad t > 1; \tag{2.23}$$

$$⨍_{-\infty}^{0} \frac{\log q'(u)}{(u-t)\sqrt{u}} du = \int_0^1 \frac{\theta_2(u)}{(u-t)\sqrt{u}} du + \int_1^{\infty} \frac{\theta_1(t)}{(u-t)\sqrt{u}} du, \qquad t < 0; \tag{2.24}$$

$$\log q_2'(t) = \frac{-\sqrt{t}}{\pi} \left\{ \int_{-\infty}^{0} \frac{\log q'(u)}{(u-t)\sqrt{-u}} du + ⨍_0^1 \frac{\theta_2(u)}{(u-t)\sqrt{u}} du + \int_1^{\infty} \frac{\theta_1(u)}{(u-t)\sqrt{u}} du \right\} \qquad 0 < t < 1; \tag{2.25}$$

$$\log q_1'(t) = \frac{-\sqrt{t}}{\pi} \left\{ \int_{-\infty}^{0} \frac{\log q'(u)}{(u-t)\sqrt{-u}} du + \int_0^1 \frac{\theta_2(u)}{(u-t)\sqrt{u}} du + ⨍_1^{\infty} \frac{\theta_1(u)}{(u-t)\sqrt{u}} dt \right\} \qquad t > 1, \tag{2.26}$$

where $⨍$ denotes a singular integral in the sense of Cauchy.

The singularities in equations (2.21) to (2.26) may be removed by using the following identities

$$⨍_0^1 \frac{1}{(u-t)\sqrt{u}}\,du = \frac{1}{\sqrt{t}}\log\left(\frac{1-\sqrt{t}}{1+\sqrt{t}}\right), \qquad 0 < t < 1; \tag{2.27}$$

$$⨍_1^\infty \frac{1}{(u-t)\sqrt{u}}\,du = \frac{1}{\sqrt{t}}\log\left(\frac{\sqrt{t}+1}{\sqrt{t}-1}\right), \qquad t > 1; \tag{2.28}$$

$$⨍_{-\infty}^0 \frac{1}{(u-t)\sqrt{u}}\,du = 0, \qquad t < 0. \tag{2.29}$$

Using (2.27) to (2.29), equations (2.21) to (2.26) take the form

$$\log q'(t) = \frac{-\sqrt{-t}}{\pi}\left\{\int_0^1 \frac{\log q_2'(u)}{(u-t)\sqrt{u}}\,du + \int_1^\infty \frac{\log q_2'(u)}{(u-t)\sqrt{u}}\,du\right\}, \qquad t < 0; \tag{2.30}$$

$$\theta_2(t) = \frac{\sqrt{t}}{\pi}\left\{\int_0^1 \frac{\log q_2'(u) - \log q_2'(t)}{(u-t)\sqrt{u}}\,du + \int_1^\infty \frac{\log q_1'(u)}{(u-t)\sqrt{u}}\,du\right\}$$
$$+ \frac{\log q_2'(t)}{\pi}\log\left(\frac{1-\sqrt{t}}{1+\sqrt{t}}\right), \qquad 0 < t < 1; \tag{2.31}$$

$$\theta_1(t) = \frac{\sqrt{t}}{\pi}\left\{\int_0^1 \frac{\log q_2'(u)}{(u-t)\sqrt{u}}\,du + \int_1^\infty \frac{\log q_1'(u) - \log q_1'(t)}{(u-t)\sqrt{u}}\,du\right\}$$
$$+ \frac{\log q_1'(t)}{\pi}\log\left(\frac{\sqrt{t}+1}{\sqrt{t}-1}\right), \qquad t > 1; \tag{2.32}$$

$$\int_{-\infty}^0 \frac{\log q'(u) - \log q'(t)}{(u-t)\sqrt{-u}}\,du = \int_0^1 \frac{\theta_2(u)}{(u-t)\sqrt{u}}\,du + \int_1^\infty \frac{\theta_1(u)}{(u-t)\sqrt{u}}\,du, \qquad t < 0; \tag{2.33}$$

$$\log q_2'(t) = \frac{-\sqrt{t}}{\pi}\left\{\int_{-\infty}^0 \frac{\log q'(u)}{(u-t)\sqrt{-u}}\,du + \int_0^1 \frac{\theta_2(u) - \theta_1(t)}{(u-t)\sqrt{u}}\,du\right.$$
$$\left. + \int_1^\infty \frac{\theta_1(u)}{(u-t)\sqrt{u}}\,du\right\} + \frac{\theta_1(t)}{\pi}\log\left(\frac{1-\sqrt{t}}{1+\sqrt{t}}\right), \qquad 0 < t < 1; \tag{2.34}$$

$$\log q_1'(t) = \frac{-\sqrt{t}}{\pi}\left\{\int_{-\infty}^0 \frac{\log q'(u)}{(u-t)\sqrt{-u}}\,du + \int_0^1 \frac{\theta_2(u)}{(u-t)\sqrt{u}}\,du\right.$$

$$+\int_1^{\infty} \frac{\theta_1(u)-\theta_1(t)}{(u-t)\sqrt{u}}\,du\Bigg\} + \frac{\theta_1(t)}{\pi}\log\left[\frac{\sqrt{t}-1}{\sqrt{t}+1}\right], \qquad t > 1. \tag{2.35}$$

3. Solution of the problem

The coordinates (x', y') of a point on the upper or lower free surface can be obtained using (2.7) and (2.10) as follows

$$z'_1(t) = (x'_0 + i) + \int_{\infty}^{t} \frac{e^{i\theta_1(u)}}{q'_1(u)} \frac{1}{\pi(1-u)}\,du, \qquad t > 1.$$

Separating real and imaginary parts we get $(x'_1(t), y'_1(t))$ for the upper free surface

$$y'_1(t) = 1 + \frac{1}{\pi}\int_{\infty}^{t} \frac{\sin\theta_1(u)}{(1-u)\,q'_1(u)}\,du, \qquad t > 1; \tag{3.1}$$

$$x'_1(t) = x'_0 + \frac{1}{\pi}\int_{\infty}^{t} \frac{\cos\theta_1(u)}{(1-u)\,q'_1(u)}\,du, \qquad t > 1. \tag{3.2}$$

For the lower free surface,

$$z'_2(t) = \int_0^t \frac{e^{i\theta_2(u)}}{q'_2(u)} \frac{1}{\pi(1-u)}\,du, \qquad 0 < t < 1.$$

Separating real and imaginary parts, we get

$$y'_2(t) = \frac{1}{\pi}\int_0^t \frac{\sin\theta_2(u)}{(1-u)\,q'_2(u)}\,du, \qquad 0 < t < 1; \tag{3.3}$$

$$x'_2(t) = \frac{1}{\pi}\int_0^t \frac{\cos\theta_2(u)}{(1-u)\,q'_2(u)}\,du, \qquad 0 < t < 1. \tag{3.4}$$

The excess pressure coefficient, C_p, at a point on the shelf, using Bernoulli's equation, is

$$C_p = 1 + \frac{2}{F^2} - q'^2, \tag{3.5}$$

where

$$C_p = \frac{p}{\frac{1}{2}\rho U^2} \tag{3.6}$$

4. Numerical solution

Integrals over infinite intervals occur in equations (2.30–2.35), but can be eliminated by making the following changes of variables:

(a) When $t < 0$, we put $t = -\cot^2\gamma, u = -\cot^2\beta$, and let $Q(\gamma) = q'(-\cot^2\gamma)$;

(b) When $0 < t < 1$, the variables t and u are left unaltered, but for uniformity we write $Q_2(t) = q'_2(t)\, \Theta_2(t) = \theta'_2(t)$;

(c) When $t > 1$, t and u are replaced by $1/t$ and $1/u$, and we put

$$Q_1(t) = q'_1\left(\frac{1}{t}\right), \Theta_1(t) = \theta'_1\left(\frac{1}{t}\right).$$

Equations (2.30–2.35) can then be written as follows:

(i) Along the lower free surface, $(0 < t < 1)$

$$Q_2(t) = \left[1 - \frac{2}{F^2}\,(Y_2(t) - 1)\right]^{1/2}, \tag{4.1}$$

$$\Theta_2(t) = \frac{\sqrt{t}}{\pi}\left\{\int_0^1 \frac{\log Q_2(u) - \log Q_2(t)}{(u-t)\sqrt{u}}\,du + \int_0^1 \frac{\log Q_1(u)}{(1-ut)\sqrt{u}}\,du\right\}$$
$$+\frac{\log Q_2(t)}{\pi}\log\left(\frac{1-\sqrt{t}}{1+\sqrt{t}}\right), \tag{4.2}$$

$$Y_2(t) = \frac{1}{\pi}\int_0^t \frac{\sin\Theta_2(u)}{(1-u)\,Q_2(u)}\,du, \tag{4.3}$$

$$X_2(t) = \frac{1}{\pi}\int_0^t \frac{\cos\Theta_2(u)}{(1-u)\,Q_2(u)}\,du; \tag{4.4}$$

(ii) Along the upper free surface: $(0 < t < 1)$

$$Q_1(t) = \left[1 - \frac{2}{F^2}(Y_1(t) - 1)\right]^{1/2}, \tag{4.5}$$

$$\Theta_1(t) = \frac{\sqrt{t}}{\pi}\left\{\int_0^1 \frac{\log Q_2(u)}{(ut-1)\sqrt{u}}\,du - \int_0^1 \frac{\log Q_1(u) - \log Q_1(t)}{(u-t)\sqrt{u}}\,du\right\}$$
$$+\frac{\log Q_1(t)}{\pi}\log\left(\frac{1+\sqrt{t}}{(1-\sqrt{t})}\right), \tag{4.6}$$

$$Y_1(t) = 1 + \frac{1}{\pi}\int_0^t \frac{\sin\Theta_1(u)}{u(1-u)\,Q_1(u)}\,du, \tag{4.7}$$

$$X_1(t) = x_0 + \frac{1}{\pi}\int_0^t \frac{\cos\Theta_1(u)}{u(1-u)\,Q_1(u)}\,du; \tag{4.8}$$

(iii) Along the solid boundary: $(0 < \gamma < \pi/2)$

$$\log Q(\gamma) = -\frac{\tan\gamma}{\pi}\left\{\int_0^1 \frac{\log Q_2(u)}{(1+u\tan^2\gamma)\sqrt{u}}\,du + \int_0^1 \frac{\log Q_1(u)}{(u+\tan^2\gamma)\sqrt{u}}\,du\right\}, \tag{4.9}$$

$$C_p = 1 + \frac{2}{F^2} - Q^2(\gamma); \tag{4.10}$$

(iv) For numerical checking:

$$2\int_0^{\pi/2} \frac{\log Q(\beta) - \log Q(\gamma)}{(\tan^2\beta - \tan^2\gamma)}\sec^2\beta\,d\beta + \int_0^1 \frac{\Theta_2(u)}{(1+u\tan^2\gamma)\sqrt{u}}\,du$$

$$+\int_0^1 \frac{\Theta_1(u)}{(u+\tan^2\gamma)\sqrt{2}}\,du = 0, \qquad 0 < \gamma < \frac{\pi}{2}. \tag{4.11}$$

Equations (4.1–4.3) and (4.5–4.7) constitute a set of nonlinear integral equations in Q_1, Q_2, Θ_1, Θ_2, Y_1 and Y_2. Numerical solutions were obtained using an iterative method as follows:

(i) Following Southwell and Vaisey [9] the gravity-free solution

$$Y_1^{(0)} = 1, \qquad Y_2^{(0)} = 0$$

was taken as an initial approximation.

(ii) The values of $Y_1^{(0)}$ and $Y_2^{(0)}$ were substituted in (4.1) and (4.5) yielding $Q_1^{(0)}$ and $Q_2^{(0)}$, which in turn were substituted in (4.2) and (4.6) yielding $\Theta_1^{(0)}$ and $\Theta_2^{(0)}$.

(iii) The values of $\Theta_1^{(0)}$, $Q_1^{(0)}$, $\Theta_2^{(0)}$ and $Q_2^{(0)}$ were substituted in (4.3) and (4.7) yielding $Y_1^{(1)}$ and $Y_2^{(1)}$, completing the first cycle of iteration.

(iv) The second (and subsequent) cycles were carried out by returning to (ii) above, and increasing the superscripts by one.

Iteration was continued until successive approximations differed by an appropriately small amount, usually from 5×10^{-5} to 5×10^{-7}. After each iteration, equation (4.11) was used as a numerical check.

When the final values of Q_1, Q_2, Θ_1, Θ_2, Y_1 and Y_2 had been obtained, equations (4.4) and (4.8) were used to determine the values of X_1 and X_2; and equations (4.9) and (4.10) to determine the value of Q and the pressure coefficient C_p along the bottom of the channel.

It was found that the iteration process began to converge after two or three cycles. It usually required *four* cycles to give the *lower* free surface to *three* decimal places and the *upper* free surface to *two* decimal places; and *seven* cycles to give *five* and *four* decimal places respectively.

5. Discussion and comparisons

The shapes of the free surfaces are shown in Figure 4 for a number of different values of the Froude number, and the values of the pressure coefficient C_p for various values of F are shown in Figure 5.

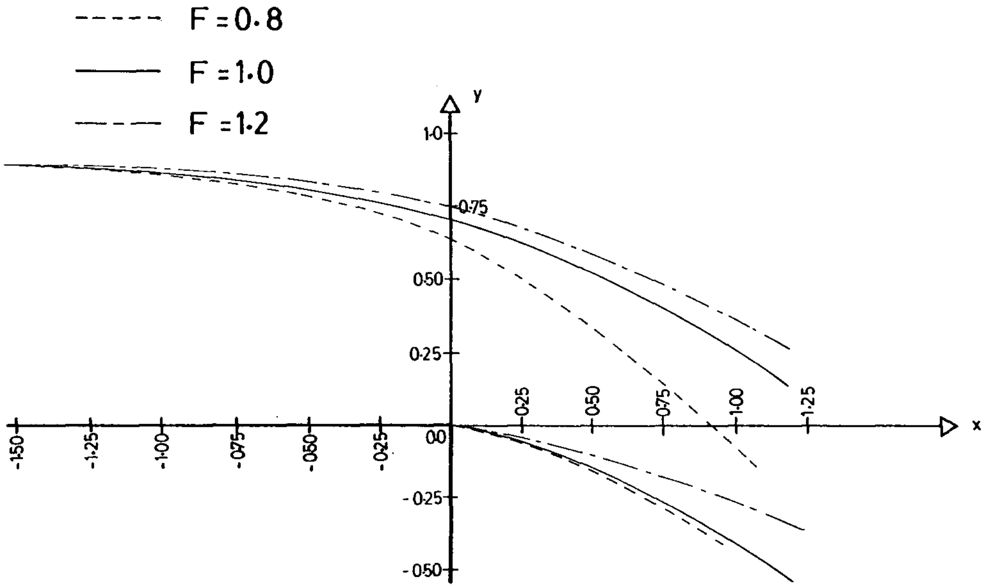

Figure 4. Waterfall profile for subcritical, critical, and supercritical approaching flows.

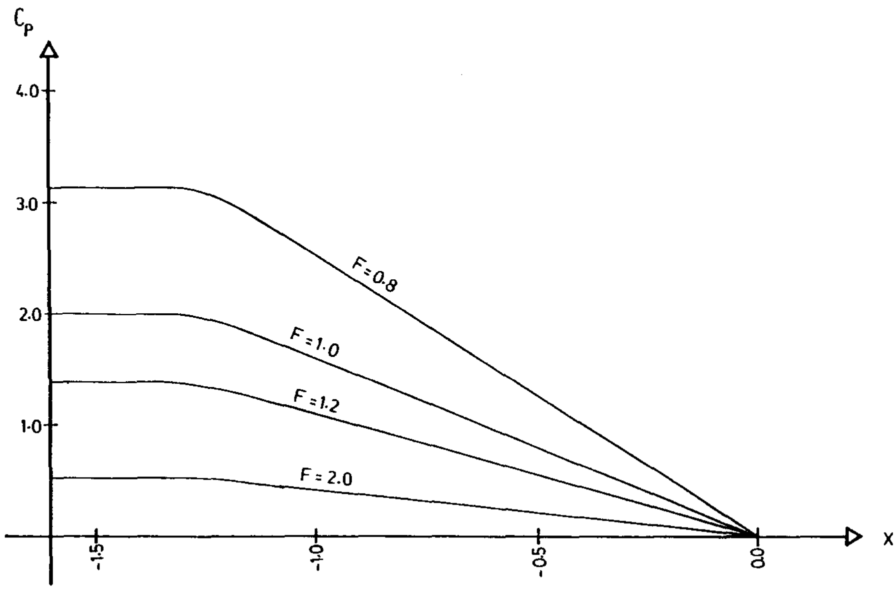

Figure 5. Cp along the flat shelf for different values of froude number.

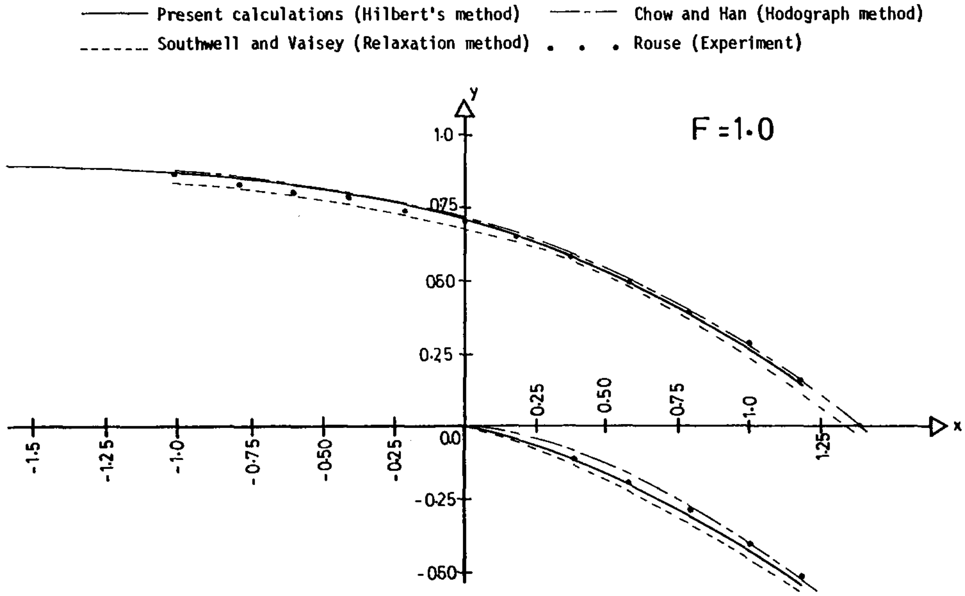

Figure 6. Comparison with experimental data and other results.

A comparison with the experimental results of Rouse [6]; the hodograph method employed by Chow and Han [1]; and the relaxation solution of Southwell and Vaisey [9] is shown in Figure 6, when the Froude number is equal to unity. It appears that the numerical results of the present paper agree closely with Rouse's experimental data.

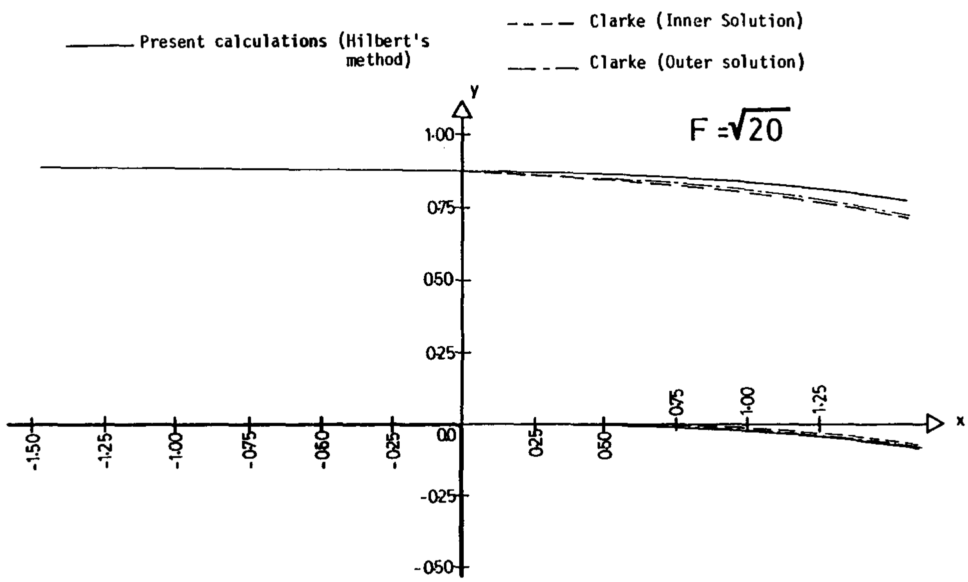

Figure 7. Comparison with perturbation method for $F=\sqrt{20}$.

As mentioned earlier, Clark [2] obtained solutions of this problem for large Froude numbers. A comparison with the numerical results of the present paper is shown in Figure 7 for $F = \sqrt{20}$. It appears that there is close agreement near the crest, but further downstream Clarke's solution yields a thinner jet.

Acknowledgements

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REFERENCES

[1] Chow, W. L. and Han, T., Inviscid solution for the problem of free overfall, *J. Applied Mech.* 46 (1979) 1–5.
[2] Clarke, N. S., On two-dimensional inviscid flow in a waterfall, *J. Fluid Mech.* 22 (1965) 359–369.
[3] Geer, J. and Keller, J. B., Slender streams, *J. Fluid Mech.* 93 (1979) 97–115.
[4] Hay, N. and Markland, E., The determination of the discharge over weirs by the electrolytic tank, *Institution of Civil Engng.* London, Proc. 10 (1958) 59–85.
[5] Keller, J. B. and Geer, J., Flows of thin streams with free boundaries, *J. Fluid Mech.* 59 (1973) 417–432.
[6] Rouse, H. and Ince, S., *History of hydraulics.* Iowa Institute of Hydraulic Research, State University of Iowa, 1957.
[7] Scarborough, J. B., *Numerical mathematical analysis,* 6th ed., John Hopkins, Baltimore, 1966.
[8] Song, S. C., A quasi-linear and linear solution for non-separated and separated two dimensional irrotational flow about lifting bodies, University of Minnesota, SAF Lab. Tech. Paper B-43, 1963.
[9] Southwell, R. V. and Vaisey, G., Relaxation methods applied to engineering problems, XII, Fluid motions characterized by 'free stream-lines', *Phil. Trans. Royal Soc., London* A240 (1946) 117–161.